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# Synthesis Technique for the Nonclassically Damped Structures Using Real Schur Vectors

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## Introduction

WITH the extensive application of composite materials and energy-dissipative devices in engineering structures, dynamic analyses of nonclassically damped systems are now frequently found. For damped systems, Foss[1] has proposed what is commonly known as the complex mode theory. Caughey and O'Kelly,[2] Liu and Wilson,[3] and Caughey and Ma[4] have given the conditions under which the damped systems can be decoupled with the real modes in physical space or configuration space. In practice, the verification of those conditions is often prohibitive due to the size of the problem and numerical difficulty. Thus, in general, the use of complex modes theory in state space is the practice. The complex mode theory is actually based on Jordan decomposition, where numerical difficulties arise under defectiveness conditions.[5,6] Zheng et al.[7] and Ren and Zheng[8,9] proposed the quasidecoupling technique, which is based on real Schur's decomposition instead of on Jordan's decomposition. Distinctive to the complex mode theory, the quasidecoupling approach involves only orthogonal operations, indiscriminately applied to defective and nondefective systems and uses only one side of the invariant space. The difference between Jordan's decomposition and Schur's decomposition lies in the adoption of the bases of the invariant subspace. The adoption of complex eigenvectors results in the complex mode theory with the diagonal form or Jordan form (defective cases) as the canonical form. On the other hand, the adoption of the real Schur vectors results in the quasidecoupling approach, with the real Schur form as the canonical form.

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Following decades of development, the mode synthesis technique is now widely used in engineering applications, with its theoretical foundation solidly established. The mode synthesis technique for nonproportionally or nonclassically damped systems has also been developed by many researchers[10–12] and may generally be classified into two categories: those that use real modes and those that use complex modes. It has been agreed that, if real modes are used in the synthesis, off-diagonal elements of the transformed damping should be reserved for better damping synthesis.[10] When using the complex mode synthesis in state space, the effects of the off-diagonal elements in the real mode case are automatically taken into consideration because of the biorthogonality of the complex constraint dynamic modes. From the present understanding of the principles of mode synthesis, it is the subspace spanned by the modes used in the synthesis that is of the essence, not the modes themselves. Based on these arguments, whereas the complex modes were used in the mode synthesis technique, the real Schur vectors spanning the same invariant subspace can be used in replacement. This Note tries to use the real Schur vectors in the mode synthesis technique for nonclassically damped systems. For conciseness and because of limitations on the size of this Note, the work is carried out in the fixed interface method.

## Substructure Analysis with Arnoldi Reduction and the Real Schur Decomposition

The proposed component mode synthesis is illustrated with two substructures. The governing equation of the damped substructures can be written as

$$\begin{bmatrix} m_{ii}^s & m_{ij}^s \\ m_{ji}^s & m_{jj}^s \end{bmatrix} \begin{bmatrix} \ddot{x}_i^s \\ \ddot{x}_j^s \end{bmatrix} + \begin{bmatrix} c_{ii}^s & c_{ij}^s \\ c_{ji}^s & c_{jj}^s \end{bmatrix} \begin{bmatrix} \dot{x}_i^s \\ \dot{x}_j^s \end{bmatrix} + \begin{bmatrix} k_{ii}^s & k_{ij}^s \\ k_{ji}^s & k_{jj}^s \end{bmatrix} \begin{bmatrix} x_i^s \\ x_j^s \end{bmatrix} = \begin{bmatrix} f_i^s(t) \\ f_j^s(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f_{12}^s(t) \end{bmatrix} \tag{1}$$

where superscript $s$ is the index of substructures and subscripts $i$ and $j$ are the indices of the variables corresponding to the internal degrees of freedom (DOFs) and interface DOFs, respectively. In Eq. (1), $m_{ab}^s$, $c_{ab}^s$, and $k_{ab}^s$ $(a, b = i, j; s = 1, 2)$ are the mass matrix, damping matrix, and stiffness matrix partitions corresponding to the internal/external DOFs. The nonsymmetric damping matrix is allowed in this research as in the case of damped structures with Coriolis effects, e.g., rotating helicopter rotor systems. Also, $f_i^s$ and $f_j^s$ are the external forces acting on the internal DOFs $x_i^s$ and the interface DOFs $x_j^s$, respectively. Here, $f_{12}^s$ is the interface force satisfying

$$f_{12}^1 + f_{12}^2 = 0 \tag{2}$$

By rewriting the governing equation (1) and the equilibrium equation (2) of the interface forces in state-space form, we have

$$\begin{bmatrix} A_{ii}^s & A_{ij}^s \\ A_{ji}^s & A_{jj}^s \end{bmatrix} \begin{bmatrix} \dot{y}_i^s \\ \dot{y}_j^s \end{bmatrix} + \begin{bmatrix} B_{ii}^s & B_{ij}^s \\ B_{ji}^s & B_{jj}^s \end{bmatrix} \begin{bmatrix} y_i^s \\ y_j^s \end{bmatrix} = \begin{bmatrix} F_i^s(t) \\ F_j^s(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F_{12}^s(t) \end{bmatrix} \tag{3a}$$

$$F_{12}^1 + F_{12}^2 = 0 \tag{3b}$$

where

$$A_{ab}^s = \begin{bmatrix} c_{ab}^s & m_{ab}^s \\ -m_{ab}^s & 0 \end{bmatrix}, \qquad B_{ab}^s = \begin{bmatrix} k_{ab}^s & \\ & m_{ab}^s \end{bmatrix}$$

$$y_a^s = \begin{bmatrix} x_a^s \\ \dot{x}_a^s \end{bmatrix}, \qquad F_a^s = \begin{bmatrix} f_a^s \\ 0 \end{bmatrix}, \qquad F_{12}^s = \begin{bmatrix} f_{12}^s \\ 0 \end{bmatrix}$$

The fixed interface modes are usually defined as one set of the independent bases of the lower invariant subspace of the interface-fixed substructure. In this Note, the orthogonal bases of the invariant subspace of the interface fixed substructures are adopted, i.e., the invariant subspace of the following eigenproblem:

$$\lambda A_{ii}^s Y_i^s + B_{ii}^s Y_i^s = 0 \tag{4}$$

where $\lambda$ and $Y_i^s$ are the substructure eigenpair. The solution procedure is outlined in the following; details may be found in Refs. 7–9.

1) Use the Arnoldi reduction algorithm[13] to generate the Arnoldi vectors $\boldsymbol{q}_k (k = 1, \ldots, K)$ (Refs. 8 and 12) and formulate the Arnoldi matrix $Q_K = [q_1, \ldots, q_K]$, satisfying

$$Q_K^T B_{ii}^s Q_K = I^s, \qquad Q_K^T A_{ii}^s Q_K = H^s$$

where $I^s$ is a unit matrix and $H^s$ is the upper Hessenberg matrix.

2) Carry out the real Schur decomposition[6,14] of $H^s$, i.e.,

$$U^T H^s U = R^s, \qquad U^T U = I^s$$

where $U$ is an orthogonal matrix and $R^s$ is the real Schur form of $H^s$.

3) Formulate

$$\phi_i^s = Q_K U$$

where a truncation is made so that all of the columns of $\phi_i^s$ span the invariant subspace corresponding to the lower spectra of eigenproblem (4) with prespecified precision.[8] From the definition of $\phi_i^s$, we have

$$\phi_i^{s^T} B_{ii}^s \phi_i^s = I^s \tag{5a}$$

$$\phi_i^{s^T} A_{ii}^s \phi_i^s = R_{ii}^s \tag{5b}$$

where $R_{ii}^s$ takes the following quasi-upper triangular form:

$$R_{ii}^s = \begin{bmatrix} r_{11}^s & r_{12}^s & \cdots & r_{1k_i}^s \\ & r_{22}^s & \cdots & r_{2k_i}^s \\ & & \ddots & \vdots \\ & & & r_{k_i^s k_i^s}^s \end{bmatrix} \tag{6}$$

where $r_{ll}^s (l = 1, \ldots, k_i^s)$ is either real scalar or a $2 \times 2$ block having complex conjugate eigenvalues. The invariant property of $\phi_i^s$ can be demonstrated by manipulating Eqs. (5a) and (5b) to obtain the following equation:

$$A_{ii}^s \phi_i^s = B_{ii}^s \phi_i^s R_{ii}^s \tag{7}$$

The generation of Arnoldi vectors is as stable as the generation of Lanczos vectors for the symmetric matrix. In addition, the real Schur decomposition of the projected upper Hessenberg matrix $H^s$ can always be carried out with the HQR3 algorithm[14] irrespective of whether the system is defective, avoiding numerically ascertaining the Jordan structure of a system, which is difficult when the system is defective.[5,6] Meanwhile, only one side of the eigensolution is involved. The constraint mode matrix in state-space mode synthesis is defined as[12]

$$\varphi^s = \begin{bmatrix} k_{ii}^{s^{-1}} k_{ij}^s & \\ & k_{ii}^{s^{-1}} k_{ij}^s \end{bmatrix} \tag{8}$$

The preceding definition of the constraint modes guarantees the consistency between the displacement and the velocity of the system. With the fixed interface real Schur vectors and the preceding constraint modes, introduce the transformation from physical coordinates to the generalized coordinates:

$$\begin{bmatrix} y_i^s \\ y_j^s \end{bmatrix} = \Phi^s \begin{bmatrix} \boldsymbol{q}^s \\ y_j^s \end{bmatrix} \tag{9}$$

where

$$\Phi^s = \begin{bmatrix} \phi_i^s & \varphi^s \\ 0 & I \end{bmatrix}$$

and $\boldsymbol{q}^s$ is the generalized coordinate vector associated with the real Schur vector. Substitute Eq. (9) into Eq. (3) and premultiply Eq. (3) with $\Phi^{s^T}$; we then have

$$\begin{bmatrix} R_{ii}^s & \bar{A}_{ij}^s \\ \bar{A}_{ji}^s & \bar{A}_{jj}^s \end{bmatrix} \begin{bmatrix} \dot{q}^s \\ \dot{y}_j^s \end{bmatrix} + \begin{bmatrix} I & \bar{B}_{ij}^s \\ \bar{B}_{ji}^s & \bar{B}_{jj}^s \end{bmatrix} \begin{bmatrix} q^s \\ y_j^s \end{bmatrix} = \begin{bmatrix} \bar{F}_i^s(t) \\ \bar{F}_j^s(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F_{12}^s(t) \end{bmatrix} \tag{10}$$

where the block elements are given by

$$\bar{A}_{ij}^s = \phi_i^{s^T} \left( A_{ii}^s \varphi^s + A_{ij}^s \right), \qquad \bar{A}_{ji}^s = \left( \varphi^{s^T} A_{ii}^s + A_{ji}^s \right) \phi_i^s$$

$$\bar{A}_{jj}^s = \phi_i^{s^T} \left( A_{ii}^s \varphi^s + A_{ij}^s \right) + \varphi^{s^T} A_{ji}^s + A_{jj}^s$$

$$\bar{B}_{ij}^s = \bar{B}_{ji}^{s^T} = \phi_i^{s^T} \left( B_{ii}^s \varphi^s + B_{ij}^s \right)$$

$$\bar{B}_{jj}^s = \phi_i^{s^T} \left( B_{ii}^s \varphi^s + B_{ij}^s \right) + \varphi^{s^T} B_{ji}^s + B_{jj}^s$$

$$\bar{F}_i^s = \phi_i^{s^T} F_i^s, \qquad \bar{F}_j^s = \varphi^{s^T} F_i^s + F_j^s$$

## Synthesis Analysis

Enforcing the equilibrium condition of the interface forces (3a), the resulting synthesized equation can be written in generalized coordinates:

$$\bar{A}\dot{Z} + \bar{B}Z = \bar{F} \tag{11}$$

$$\bar{A} = \begin{bmatrix} R_{ii}^1 & 0 & \bar{A}_{ij}^1 \\ 0 & R_{ii}^2 & \bar{A}_{ij}^2 \\ \bar{A}_{ji}^1 & \bar{A}_{ji}^2 & \sum_{s=1}^{2} \bar{A}_{jj}^s \end{bmatrix}, \qquad Z = \begin{bmatrix} q^1 \\ q^2 \\ y_j \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} I^1 & 0 & \bar{B}_{ij}^1 \\ 0 & I^2 & \bar{B}_{ij}^2 \\ \bar{B}_{ji}^1 & \bar{B}_{ji}^2 & \sum_{s=1}^{2} \bar{B}_{jj}^s \end{bmatrix}, \qquad \bar{F} = \begin{bmatrix} \bar{F}_i^1 \\ \bar{F}_i^2 \\ \sum_{s=1}^{2} \bar{F}_j^s \end{bmatrix}$$

Again, the Arnoldi reduction and the generalized real Schur decomposition can be used in combination[8] to solve a set of the $\bar{B}$-orthogonal basis for the invariant subspace corresponding to the lower eigenvalues of the reduced system; i.e.,

$$\phi_Z^T \bar{A} \phi_Z = R_Z, \qquad \phi_Z^T \bar{B} \phi_Z = I_Z$$

where $R_Z$ is in upper quasitriangular form as is $R_{ii}^s$ in Eq. (5b). $I_Z$ is a unit matrix. The eigenvalues of the original system can be obtained by the inverse of the eigenvalues of the diagonal blocks of $R_Z$. With a real Schur vector of the whole system, the dynamic response problem can be solved; details may be found in Refs. 7–9.

## Illustrative Example

A beam with concentrated damping is shown in Fig. 1. The material and section properties are $E = 2.1 \times 10^{11}$ Pa, $\upsilon = 0.3$, $J = 1.167 \times 10^{-5}$ m$^4$, line mass density $\rho_l = 11.3$ kg/m, and inertial moment per unit length $I = 5.679 \times 10^{-3}$ kg/m. The beam is axially rigid. The damping coefficient of each damper $c = 2.0 \times 10^2$ Nm/s.

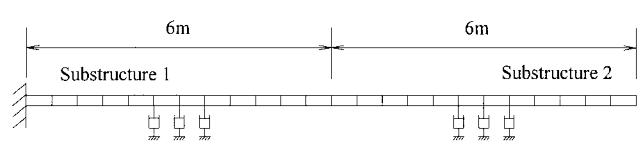


**Fig. 1 Finite element method mesh and the substructure partition of the beam.**

**Table 1 Results on the clamped–free beam**

| | Present solution | | Solution on whole model | |
|---|---|---|---|---|
| Mode | Damping | Frequency | Damping | Frequency |
| 1 | 3.396 | 72.299 | 3.396 | 72.298 |
| 2 | 8.768 | 199.263 | 8.768 | 199.263 |
| 3 | 7.552 | 390.858 | 7.541 | 390.769 |
| 4 | 2.806 | 646.240 | 2.804 | 646.181 |
| 5 | 3.177 | 966.672 | 3.101 | 965.252 |
| 6 | 5.256 | 1349.290 | 5.142 | 1348.080 |

**Table 2 Results on the clamped–clamped beam**

| | Present solution | | Solution on whole model | |
|---|---|---|---|---|
| Mode | Damping | Frequency | Damping | Frequency |
| 1 | 3.949 | 10.676 | 3.949 | 10.676 |
| 2 | 1.880 | 71.222 | 1.880 | 71.221 |
| 3 | 7.217 | 199.288 | 7.218 | 199.286 |
| 4 | 6.774 | 390.919 | 6.757 | 390.736 |
| 5 | 2.608 | 646.301 | 2.587 | 646.120 |
| 6 | 3.108 | 968.164 | 3.110 | 965.132 |

The beam is partitioned into two substructures; both are discretized with 12 equidistant beam elements. Two computations are performed for the clamped–free and the clamped–clamped boundary conditions, respectively. Comparison between the present solutions and the solutions on the whole model are shown in Tables 1 and 2. Eight fixed interface Schur vectors are used for the substructure, corresponding to four pairs of complex eigenvalues. The original and the reduced model sizes (state variables) are 144 and 22 for the clamped–free model and 138 and 22 for the clamped–clamped model, respectively.

## Conclusion

Real Schur vectors are adopted as the fixed interface principal modes, and a mode synthesis technique is carried out in state space for nonclassically damped structures involving only real and orthogonal operations. Compared with the complex mode synthesis, the proposed method has features such as completely real operations, uniform application to both the nondefective and defective systems, and the requirement of only one side of the invariant subspace. The resulting advantages stem from the replacement of the Jordan decomposition with the real Schur decomposition as the canonical form in the analysis.

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